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STRESS FIELD OF INCOMPLETE SHEAR AND COMBINATIONS OF INCOMPLETE SHEARS DUE TO ELASTIC INTERACTION

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A stress field of incomplete shear and effects due to elastic interaction of incomplete shears are considered.

Definition of Incomplete Shear and Relation with Dislocations. Incomplete shear (ISh) is a portion of a plane of shear in which the shear displacements are larger than in the rest of the plane, where the state is considered elastic. The concept of ISh is used to describe the plastic deformation of elastic bodies. One-dimensional crystalline models of ISh for its edge in the direction of shear (the Frenkel'–Kontorova and Kosevich models of an edge dislocation) [1] lead to a symmetric distribution of shear displacements of atoms relative to the edge of ISh. This distribution is used as the boundary condition in solution of a two-dimensional elastic problem, and it predetermines an elastic field that is symmetric about the line that passes through the atomic extraplane (the Peierls–Nabarro model of an edge dislocation) [1]. In [2], the field of ISh is found by direct solution of a plane elastic problem (without using results of the one-dimensional model). This field differs from the Peierls–Nabarro field and, in particular, does not exhibit the symmetry indicated above. In [2], mass transfer by plastic shears is considered and the distribution of only relative dilatation (hydrostatic pressure) in the elastic field of ISh is shown.

According to the Weingarten theorem [3], if no special assumptions on the shape of a cut are adopted, Volterra dislocations can be generated only in a multiply connected body and only in the case where the cut whose edges undergo rigid displacements reduces the connectivity of the body. ISh is produced in a simply connected body, and, hence, it leads to formation of a Somigliana's dislocation. It is natural to believe that in crystals, i.e., in bodies that exhibit translational symmetry, the relative displacements of points at the edges of a cut are equal to the translation vector, and, hence, $u = \text{const}$ on a considerable portion of the cut length. Therefore, crystal dislocations are usually modeled by Volterra dislocations. However, in many cases, there is no reason to assume that the relative displacements of points at the edges of a cut are constant quantities (the edges of the cut move as a rigid body). Among these cases are shears along grain boundaries, interfaces, in amorphous bodies, and along surfaces of contact of solids, i.e., the cases where the translational symmetry on the surfaces of a cut is broken. Indeed, by the adopted definition of ISh, the displacements in an ISh region are not related to the translation vector, and, hence, atomic correspondence is not assumed in the ISh region. The concept of ISh can be used to describe shears in crystals if $u \neq \text{const}$, for example, when a shear is related to a large number of translational dislocations or interlayers of twins or martensite phases whose thickness is not constant.

The magnitude of displacements in an ISh region and, hence, the field related to the ISh, depend on the external field. Residual fields are described in terms of dislocations. Plastic deformation is determined by the fields acting in loaded bodies. The choice of incomplete shears instead of dislocations as structural elements allows one to describe fields both in unloaded and loaded bodies.

Thus, there are two reasons why ISh should be studied: the novelty of the elastic field and new regions of application. In the present paper, we study an elastic field of ISh for a plane model in an external stress field

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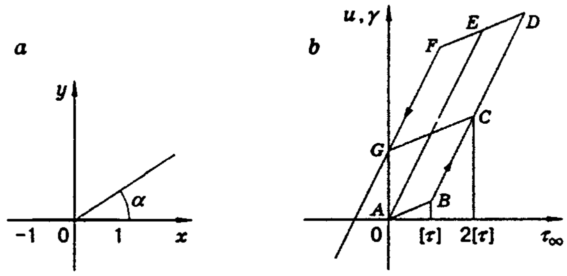

Fig. 1

that becomes a homogeneous shear at infinity, and also the effects due to the elastic interaction of incomplete shears.

Formation ISh and Its Contribution to Macrodeformation. We consider an infinite plane in an external homogeneous shear stress field with maximum tangential stresses τ_∞ directed at angles α and $\alpha+\pi/2$ to the x axis (Fig. 1a, here and below, ISh regions are shown by thick segments). Let $[\tau]$ be the critical resistance to shear in the region $-1 < x < 1$, $y = 0$, and $\alpha = 0$. As long as $\tau_\infty \leqslant [\tau]$, the deformation is elastic (segment AB in Fig. 1b, which shows the projection of the shear displacement $u \equiv u_x$ at the coordinate origin and the relative macroshear strain γ versus stress τ_∞). When the condition

$$\tau_\infty > [\tau] \tag{1}$$

is satisfied with increase in τ_∞, the shear displacements in the region grow and become large (plastic) compared to the elastic displacements in the rest of the plane. ISh is produced. The macroscopic plastic shear strain due to microshear with a uniform displacement in the entire shear region b is

$$\gamma_{\text{plast}} = b\lambda/L^2, \tag{2}$$

where λ is the size of the shear region and L is the size of the specimen. The distribution of large displacements of points adjacent to the ISh region on one side relative to points adjacent to the ISh region on the other side [2] is described by

$$u(x) = \frac{2(1-\nu)}{\mu}\sqrt{(l^2-x^2)}(\tau_\infty - [\tau]). \tag{3}$$

Here ν is the Poisson ratio, μ is the shear modulus, $2l$ is the length of the ISh region; τ_∞ should be considered a tangential stress that would operate in the ISh region without formation of ISh. From (2) and (3) we obtain

$$\gamma_{\text{plast}} = \int\limits_{-l}^{l} u(x)L^{-2}dx = \frac{\pi(1-\nu)}{\mu L^2}(\tau_\infty - [\tau])l^2. \tag{4}$$

The dependence of $u(x)$ on τ_∞ (3) and the dependence of γ_{pl} on τ_∞ (4) are linear in the plastic region (segment BD in Fig. 1b).

The elastic field of ISh, tending to unload, produces fictitious reverse stresses τ_{fict} in the ISh region. The equilibrium condition has the form

$$\tau_\infty - [\tau] - \tau_{\text{fict}} = 0. \tag{5}$$

During unloading, τ_{fict} tends to produce reverse displacements in the ISh region. The sign of $[\tau]$ changes, and this prevents reverse motion. If $|\tau_\infty| \leqslant 2[\tau]$, then $|\tau_{\text{fict}}|$ does not exceed $[\tau]$ and plastic displacements remain (polygonal line BCG). If $\tau_\infty > 2[\tau]$, then $|\tau_{\text{fict}}| > [\tau]$ and plastic displacements decrease (polygonal line $BDFG$) until new equilibrium is attained: $[\tau] + \tau_{\text{fict}} = 0$. The remaining defect is a Somigliana's dislocation, because it can be produced by generating these dislocations: by cutting along the ISh region, shear displacement of the cut edges, attachment of the edges, and elimination of external forces. If $[\tau] = 0$, ISh disappears during unloading and, hence, the field of ISh is entirely a singularity of the external field.

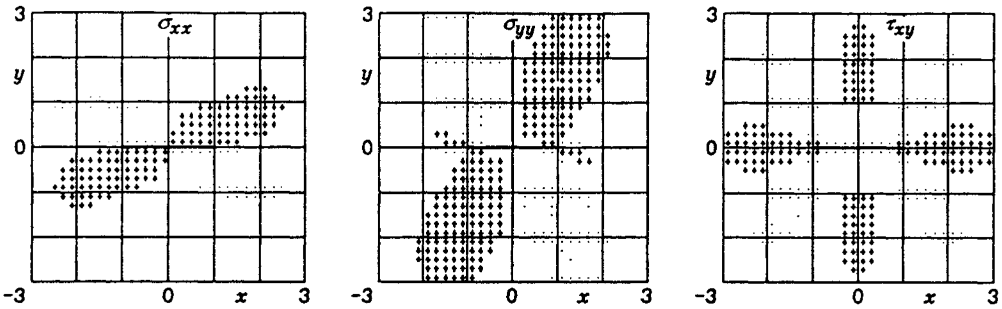

Fig. 2

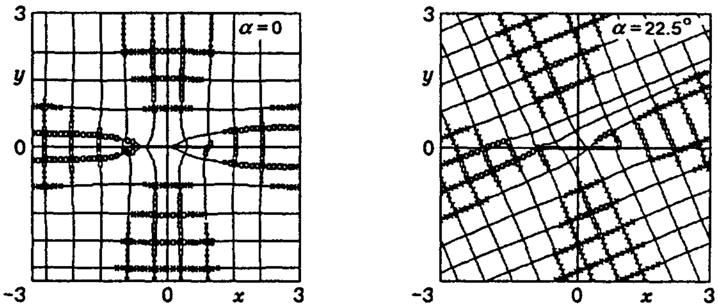

Fig. 3

According to [2], the resulting field is

$$\sigma_{ij} = (\sigma_{ij})_\infty + (\sigma_{ij})_{\text{cut}}, \tag{6}$$

where $(\sigma_{ij})_\infty$ is the external homogeneous shear field, $(\sigma_{ij})_{\text{cut}}$ is the field of the cut in the ISh region whose edges are exposed to uniform tangential stresses $\tau_{\text{cut}} = \tau_\infty - [\tau]$. These stresses are directed so as to increase the displacements produced by the external load. Since, in reality, there are no external loads at the cut edges, the increase in displacements along the cut corresponds to weakening of the interaction of the cut edges. As a result, the tangential stresses in the ISh region remain equal to $[\tau]$ as shear develops. For the homogeneous shear field adopted here, the first term in (6) is proportional to τ_∞, and the second is proportional to $(\tau_\infty - [\tau])$. Therefore, the contribution of the second term, which describes the stress-field singularities related to ISh, increases with increase in τ_∞. The ISh model considered is valid if

$$\tau_\infty - [\tau] = \tau_{\text{cut}} = \text{const.} \tag{7}$$

In what follows, it is assumed that $l = 1$, and the stresses are referred to $[\tau]$. For the stresses themselves, the notation is such that $[\tau] = 1$. Then, for example, $\tau_\infty/[\tau] = \tau_\infty$.

Elastic Field of ISh. The distribution of the stress-tensor components of the second term in (6) for $\tau_{\text{cut}} = 1$ is shown in Fig. 2 [regions in which $\sigma_{xx} < -0.125$ (compression), $\sigma_{yy} < -0.0625$ and $\tau_{xy} > 0.05$ are denoted by crosses and regions in which $\sigma_{xx} > 0.125$, $\sigma_{yy} > 0.0625$, and $\tau_{xy} < -0.05$ are denoted by dots). Note that the field in Fig. 2 gives a qualitative illustration of both the singularities of the external field near the ISh and the residual field of the Somigliana's dislocation formed at the site of ISh.

In the present paper, we consider plastic deformation that occurs by shear mechanisms. Therefore, the distribution of τ_{xy} is the determining factor. The influence of ISh can manifest itself as changes in the tangential stresses and the direction of lines of maximum tangential stresses. The run of lines of maximum tangential stresses for $\tau_\infty = 20$, i.e., at high overstresses, at which the contribution of the ISh singularities to the resulting field is significant, is shown in Fig. 3 for $\alpha = 0$ and 22.5° (portions of the lines in which $1 < \tau_{\max} < 1.05$ are marked by crosses and those in which $\tau_{\max} > 1.05$ are marked by rectangles). Outside of

TABLE 1

No.	Subsequent ISh	k	Combination formed
	In the field of ISh $(x, 0, 0)$		
1	$(x, 2.4, 0)$	1.150	—
2	$(y, 0, 1.9)$	1.049	Chain along Oy
3	$(y, 2.1, 0)$	1.037	Chain along Ox
4	$(x, 0, 1.9)$	1.030	Packet
	In the field of ISh $(x, 0, 0)$ and $(y, 0, 2)$		
5	$(y, 0, -2)$	1.076	Chain along Oy
6	$(x, 0, 4)$	1.056	The same
7	$(x, 0, -1.7)$	1.054	Packet
8	$(x, 2, 2.2)$	1.023	Grid
9	$(y, 3, 0)$	1.019	»
	In the field of a chain of ISh $(y, 0, 2)$, $(x, 0, 0)$, and $(y, 0, -2)$		
10	$(x, 0, 4)$	1.065	Chain along Oy
11	$(y, 4.7, 0)$	1.016	Grid
	In the field of a packet of ISh $(x, 0, 0.5)$ and $(x, 0, -0.5)$		
12	$(y, 0, 2.4)$	1.083	Chain along Oy
13	$(x, 2.6, 0)$	1.060	—
14	$(y, 2.1, 0)$	1.054	Chain along Ox
15	$(x, 0, 2.4)$	1.053	Packet

Note. For No. 1, $k(x)$ does not have a maximum and k decreases with increase in x.

the circular region adjacent to ISh and having a somewhat greater radius than l, the directions of the maximum tangential stress curves are practically the same as in the field before formation of ISh. The plane outside of the indicated region can be divided into eight approximately equal sectors with a center at the coordinate origin. Sectors with increased and decreased tangential stresses alternate. Thus, the field singularities due to ISh do not change the directions of regions of subsequent ISh but only specify their locations. The most probable secondary ISh regions are located parallel to the same mutually perpendicular directions along which primary ISh are most probable. The centers of secondary ISh regions are shifted from the centers of primary-shear regions along the lines of maximum tangential stresses, i.e., along the same mutually perpendicular directions.

Combinations of ISh. In subsequent calculations, we determined the most probable spacing between the centers of ISh and estimated the probabilities of various combinations of ISh. The case $\alpha = 0$ is studied in detail. To determine the probabilities, we used the three methods described below. The *force* method implies that the more rigorous inequality (1), the more probable the formation of ISh (however, the formation ISh is due not only to determined factors but also to random factors). Random factors were taken into account using the *energy* method, which implies that those ISh are probable for which the development is justified energetically. The *synergy* method implies that those combinations of ISh are preferred in which there are positive force relations and feedbacks between ISh.

The field in Fig. 3 for $\alpha = 0$ determines two possible schemes of interaction of ISh: a scheme in which ISh are mutually amplified if secondary ISh are located in regions with increased tangential stresses (the sectors containing the coordinate axes), and a scheme in which ISh are mutually weakened if secondary ISh are located in the rest of the plane (the circular region and diagonal sectors). The first scheme is more probable than the second by all three methods.

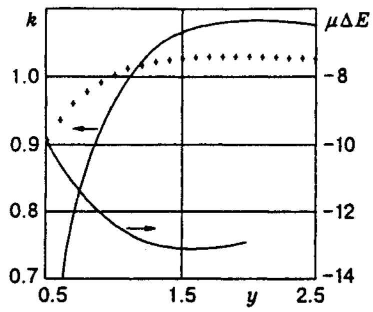

Fig. 4

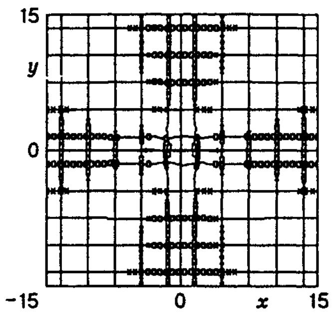

Fig. 5

In the calculations, we assumed that the lengths of ISh regions are identical and equal to 2. To designate ISh, we used the notation of the axis to which the shear region is parallel and the notation of the coordinates of the center of the ISh region; for example, for the ISh region in Fig. 1a, we write $(x, 0, 0)$. In the calculations, we specified a number of ISh regions that are parallel to the maximum tangential stress curves, calculated the coefficient $k = \langle\tau_{\max}\rangle/\tau_\infty$, where $\langle\tau_{\max}\rangle$ is the average maximum tangential stress in the regions of possible location of ISh, and selected the region for which the value of k is maximal [in Fig. 4 the crosses shows the dependence of k versus y for ISh $(x, 0, y)$ in the field of ISh $(x, 0, 0)$ at $\tau_\infty = 3$]. The coefficient k characterizes the effect of a primary ISh on a secondary ISh. Then, we calculated the k of the reverse effect of the secondary ISh on the primary ISh. The resulting fields were found with allowance for both coefficients. In the fields of mutual amplification of ISh, $k > 1$.

The maximum tangential stresses acting in subsequent ISh regions differ by not more than 0.05 from the average value of this stress for the corresponding region. Ignoring these deviations, we assumed that condition (7) is satisfied and used the ISh model described above to calculate the fields of combinations of ISh.

Table 1 gives the initial combinations of ISh, the coefficients k for subsequent ISh, and the combinations of ISh formed.

Calculation results (see Table 1) show that the development of the initial ISh (No. 1) is most probable. This shear region grows through the body. After that, the stress field becomes homogeneous, and the process is repeated. The second most probable scheme involves formation of a chain of ISh with mutually perpendicular regions whose centers are displaced from one another by about 2 along one of the coordinate axes (Nos. 2, 3, 5, 6, and 10). With increase in the number of ISh in the chain, the coefficient k somewhat increases (Nos. 2 and 10). In addition, the probability of formation of a packet, i.e., a combination of ISh consisting of several closely spaced ISh with parallel regions, increases (Nos. 4 and 7). With formation of packets, k increases. Consequently, the direct relationships and feedbacks between ISh are enhanced and the probability of regular arrangement of ISh increases. This is evident from comparison of the values of k for Nos. 2 and 12, 3, and 14 and the curve shown by crosses in Fig. 4 and the solid curve that characterizes k for ISh $(x, 0, y)$ in the field of a packet consisting of three ISh [$(x, 0, -0.4)$, $(x, 0, 0)$, and $(x, 0, 0.4)$] for $\tau_\infty = 3$ in the same figure.

The next probable scheme is a combination ISh in which the centers of ISh are located at nodes of an approximately square grid and the nearest ISh regions are mutually perpendicular. The formation of ISh chains elongated along diagonals that bisect the angle between the coordinate axes is least probable (Nos. 8, 9, and 11). In this case, chains of ISh are located on one side of the diagonal. Compared to such a chain, the formation of a square grid of ISh is more probable, because in the grid, the mutual positive relationships between ISh are stronger.

Figure 4 shows a curve of $\mu\Delta E(y)$, where ΔE is the change in the elastic energy of a square with center at the coordinate origin and side equal to 6 due to formation of a secondary ISh $(x, 0, y)$ in the field of the starting ISh $(x, 0, 0)$ for $\tau_\infty = 3$. To simplify treatment of the results, it was assumed that the system is closed, and, hence, the points at which external loads are applied are motionless and the total strain, which is equal to the sum of the elastic and plastic strains, is fixed:

$$\gamma = \gamma_{\text{elast}} + \gamma_{\text{plast}} = \text{const} \tag{8}$$

[γ_{plast} was calculated from formula (4)]. The scattered energy Q was calculated as the work on overcoming $[\tau]$:

$$Q = \int_{-l}^{l} u(x)[\tau]\, dx = \frac{\pi(1-\nu)}{\mu}(\tau_\infty - [\tau])[\tau]l^2. \tag{9}$$

The fact that the elastic energy decreases by an amount larger than Q ($Q\mu = 4.4$) indicates the possibility of formation of ISh in a region in which $k < 1$.

Combinations similar to those described above were obtained by modeling ISh by Volterra dislocations [2] for which $u \simeq$ const on 0.9 of the length of the ISh region. The only difference is that the chains along the x axis become more probable than the chains along the y axis.

Combinations of Chains of ISh. Figure 5 shows curves of maximum tangential stresses and the distribution of these stresses in the field of the most probable combination of ISh — a chain consisting of five ISh [$(x, -4, 0)$, $(y, -2, 0)$, $(x, 0, 0)$, $(y, 2, 0)$, and $(x, 4, 0)$] for $\tau_\infty = 3$ (regions with $3 < \tau_{\max} < 3.015$ are shown by crosses and regions with $\tau_{\max} > 3.015$ are shown by rectangles). The similarity of the distributions in Figs. 3 and 5 for $\alpha = 0$ outside of the regions adjacent to ISh suggests that if a chain of ISh is taken as the element, the probable combinations of these elements will be the same as for individual ISh. The indicated similarity between the configurations of fields and the combinations of shears provided that shears initially develop at smaller scale levels and then at large scale levels makes it possible to describe plastic deformation in terms of fractal structures.

During plastic deformation, besides the ISh regions considered, ISh form in regions having smaller dimensions, and diffusion mass transfer develops. These processes change the elastic field described by (6) and are not taken into account in our analysis.

Comparison with Experiment. The value of the left side of inequality (1) depends on the stress field, and the value of its right side is influenced by the anisotropy of the crystal. In crystals there are crystallographic planes and directions in them (twinning systems, easy-slip systems, and martensite transformations) for which $[\tau]$ are lower than in other planes and directions. When the anisotropy is pronounced, small departures from the orientation of elements of the systems cause a sharp increase in $[\tau]$. Therefore, we sought confirmation of the results in experimental manifestations of the established regularities of the mutual location of ISh and not in coincidence of the experimentally observed ISh regions with curves of maximum tangential stresses.

Mogilevskii [4] observed a number of such regularities in shock-wave deformation of zinc single crystals. Under these conditions, a plane deformation scheme is realized. In the plane of strain that is parallel to the wave front, a homogeneous shear stress field acts (ignoring fields of ISh). The stress level increases with approach of the shock wave. The projections of displacements in twinning planes onto the plane of strain are different from zero. Therefore, the two-dimensional scheme adopted in the analysis describes the situation in the plane of strain. The short duration of loading eliminates the course of relaxation processes by the mechanisms of diffusion and dislocation ISh of a small scale level, so that Eq. (6) is valid. Owing to the pulsed character of loading, the acting stresses far exceed $[\tau]$, and, hence, the effect of singularities of the fields generated by ISh on the development of the deformation process is increased. Thus, Mogilevskii [4] reproduced conditions that are close to those adopted in our analysis.

In [4], the predominant deformation mechanism was twinning. The twins in deformed specimens were treated as ISh regions.

Mogilevskii described [4] the case of "parquet" twinning. Here, of the six possible systems, two twinning systems act. The angle between the planes of these systems is close to 90° (the angle between the crystallographic planes of "parquet" twins is 93.8°). The linear dimensions of twins of different systems are roughly identical. Twins of each system form packets, and packets of different systems are located in checkered order, and their centers form a square grid. These regularities agree with the results of our analysis.

Mogilevskii [4] observed twinning in planes in which the tangential stresses calculated ignoring ISh fields were absent; this effect was explained [4] by the stress fields generated by shear displacements along twinning interlayers. This is also in agreement with our concepts.

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